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Bounds on the index of the signless Laplacian of a graph

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ABSTRACT

Let $G=(V,E)$ be a simple, undirected graph of order n and size m with vertex set V, edge set E, adjacency matrix A and vertex degrees $\Delta=d_1\geq d_2\geq\cdots\geq d_n=\delta$. The average degree of the neighbor of vertex v_i is $m_i=\frac{1}{d_i}\sum_{j=1}^{n}a_{ij}d_j$. Let D be the diagonal matrix of degrees of G. Then $L(G)=D(G)-A(G)$ is the Laplacian matrix of G and $Q(G)=D(G)+A(G)$ the signless Laplacian matrix of G. Let $\mu_1(G)$ denote the index of $L(G)$ and $q_1(G)$ the index of $Q(G)$. We survey upper bounds on $\mu_1(G)$ and $q_1(G)$ given in terms of the d_i and m_i, as well as the numbers of common neighbors of pairs of vertices. It is well known that $\mu_1(G)\leq q_1(G)$. We show that many but not all upper bounds on $\mu_1(G)$ are still valid for $q_1(G)$.

1. Introduction

Let $G=(V,E)$ be a simple undirected graph with n vertices $v_i\in V$ and m edges $\{v_i,v_j\}\in E$, for $i,j=1,2,\ldots,n$ and $i\neq j$. When v_i is adjacent to v_j, we denote this fact by $v_i\sim v_j$. The vertex degree of v_i is d_i, and the degree sequence of G is $d(G)=(d_1,d_2,\ldots,d_n)$, where, possibly after relabeling, $\Delta=d_1\geq d_2\geq\cdots\geq d_n=\delta$. The average degree of G is $\overline{d}=\frac{\sum_{i=1}^{n}d_i}{n}$ and the average degree of the neighbors of v_i is $m_i=\frac{1}{d_i}\sum_{v_j\sim v_i}d_j$. The eigenvalues of G are the eigenvalues of the adjacency matrix $A(G)$, given as $\lambda_1\geq\cdots\geq\lambda_{n-1}\geq\lambda_n$, where, λ_1 is called the index of G. Consider $D(G)$ as the diagonal matrix of vertex degrees of G. The *Laplacian matrix* of G is $L(G)=D(G)-A(G)$, its eigenvalues are displayed as $\mu_1\geq\cdots\geq\mu_{n-1}\geq\mu_n$ and μ_1 is the index of Laplacian matrix. Since $L(G)$ and $A(G)$ are well known, there are many results on their spectra, see e.g. [2,10,14].

The matrix $Q(G)$ was introduced in the classical book of Cvetković, Doob and Sachs on "Spectra of Graphs" [5], but without a name being given to it at that time. Later it was called "quasi-Laplacian matrix" (essentially by Chinese researchers) and more recently "signless Laplacian" [4,6]. As a consequence, papers in which $Q(G)$ plays a role belong to several categories:

(i) papers explicitly devoted to $Q(G)$, called by either of its two names;
(ii) papers devoted wholly or in part to $Q(G)$, without giving it a name;
(iii) papers in which $Q(G)$ plays a role (which may be an essential one in some proofs) but is not studied *per se*.

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Several researchers have observed that

$$\mu_1(G) \le q_1(G). \tag{1}$$

Moreover, Yan [18] derived from the Courant–Weyl inequalities the relation

$$2\lambda_1(G) \le q_1(G) \tag{2}$$

where λ_1 is the index of the adjacency matrix of G. These relations immediately imply that any lower bound on $\mu_1(G)$ is a valid lower bound on $q_1(G)$ and that doubling any lower bound on $\lambda_1(G)$ also yields a valid lower bound on $q_1(G)$. For instance, the relation $\lambda_1(G) \ge \overline{d}$ [5] implies $q_1(G) \ge 2\overline{d} \ge 2\delta$. Similarly, upper bounds on $q_1(G)$ yield valid upper bounds on $\mu_1(G)$ and $\lambda_1(G)$. But when are upper bounds on $\mu_1(G)$ or twice upper bounds on λ_1 valid bounds on $q_1(G)$?

The purpose of the present paper is to answer that question for a series of bounds on $\mu_1(G)$ expressed in terms of d_i and m_i as well as the number of common neighbors for pairs of vertices. Bounds which remain valid are considered in Section 2 and bounds which do not in Section 3.

2. Bounds on $\mu_1(G)$ in terms of vertex degrees and average degrees of neighbors valid for $q_1(G)$

A number of upper bounds on μ_1 given as functions of the degree and of the average degree of the neighbors of a vertex have been proposed in the literature. Brankov et al. [3] gathered some of them and classified them into two classes:

(1) Upper bounds depending on the vertices v_i, the d_i and m_i:

$$\mu_1 \le \max_{v_i} f(d_i, m_i).$$

These bounds are:

$$\mu_1(G) \le \max\{2d_i | v_i \in V(G)\}, \tag{3}$$

$$\mu_1(G) \le \max\{d_i + m_i | v_i \in V(G)\}, \quad [15] \tag{4}$$

$$\mu_1(G) \le \max\left\{d_i + \sqrt{d_i m_i} | v_i \in V(G)\right\}, \quad [19] \tag{5}$$

$$\mu_1(G) \le \max\left\{\sqrt{2d_i(d_i+m_i)} | v_i \in V(G)\right\}, \quad [12] \tag{6}$$

$$\mu_1(G) \le \max\left\{\frac{d_i + \sqrt{(d_i)^2 + 8(d_i m_i)}}{2} | v_i \in V(G)\right\} \quad [11]. \tag{7}$$

(2) Upper bounds depending on the edges $\{v_i, v_j\}$, the d_i and d_j, as well as the m_i and m_j:

$$\mu_1 \le \max_{v_i \sim v_j} f(d_i, m_i, d_j, m_j).$$

Examples of such bounds are:

$$\mu_1(G) \le \max_{v_i \sim v_j}\{d_i + d_j\}, \quad [1] \tag{8}$$

$$\mu_1(G) \le \max_{v_i \sim v_j}\left\{\frac{d_i(d_i+m_i) + d_j(d_j+m_j)}{d_i + d_j}\right\}, \quad [13] \tag{9}$$

$$\mu_1(G) \le \max_{v_i \sim v_j}\left\{\sqrt{d_i(d_i+m_i) + d_j(d_j+m_j)}\right\}, \quad [19] \tag{10}$$

$$\mu_1(G) \le \max_{v_i \sim v_j}\left\{2 + \sqrt{d_i(d_i+m_i-4) + d_j(d_j+m_j-4) + 4}\right\}, \quad [19] \tag{11}$$

$$\mu_1(G) \le \max_{v_i \sim v_j}\left\{\frac{(d_i+d_j) + \sqrt{(d_i-d_j)^2 + 4m_i m_j}}{2}\right\} \quad [7,20]. \tag{12}$$

Remark. It is easy to see that the bound given by the inequality (5) is better than that given by (6).

Let us consider first the series of bounds (3)–(7). The upper bound (3) was proved to be valid for $q_1(G)$ by several researchers e.g. [6], and the bound (4) also by Das [8] in a section devoted to the signless Laplacian without giving it a name. Based on a simple technique suggested in [3] we next prove that the bound (7) is also valid for $q_1(G)$.

Theorem 1. *Let G be a simple and connected graph. Then*

$$q_1(G) \le \max_{v_i} 2d_i \tag{13}$$

and

$$q_1(G) \le \max_{v_i} \frac{d_i + \sqrt{(d_i)^2 + 8(d_i m_i)}}{2}. \tag{14}$$

Proof. Let $x = (x_1, x_2, \dots, x_n)$ be a non-negative eigenvector corresponding to the eigenvalue q_1. So,

$$q_1 x = Q(G)x = Dx + Ax.$$

Let $x_i = \max_{1 \le j \le n}\{x_j\}$. Then,

$$q_1 x_i = d_i x_i + \sum_{v_j \sim v_i} x_j \le d_i x_i + \sum_{v_j \sim v_i} x_i = 2 d_i x_i$$

from where the first bound follows.

Similarly, we have

$${q_1}^2 x = Q^2 x = (D + A)^2 x = D^2 x + DAx + ADx + A^2 x,$$

from where it follows that

$${q_1}^2 x_i = (d_i)^2 x_i + d_i \sum_{v_j \sim v_i} x_j + \sum_{v_j \sim v_i} d_j x_j + \sum_{v_j \sim v_i} \sum_{v_k \sim v_j} x_k.$$

In order to prove the inequality (14), we consider a simple quadratic function of q_1:

$$({q_1}^2 + bq_1)x = (D^2 x + DAx + ADx + A^2 x) + b(Dx + Ax).$$

Since $x_i = \max_{1 \le j \le n}\{x_j\}$, it follows that

$$({q_1}^2 + bq_1)x_i = (d_i)^2 x_i + d_i \sum_{v_j \sim v_i} x_j + \sum_{v_j \sim v_i} d_j x_j + \sum_{v_j \sim v_i} \sum_{v_k \sim v_j} x_k + b\left(d_i x_i + \sum_{v_j \sim v_i} x_j\right).$$

It is easy to see that the inequalities below are true

$$d_i \sum_{v_j \sim v_i} x_j \le {d_i}^2 x_i; \qquad \sum_{v_j \sim v_i} x_j \le d_i x_i; \qquad \sum_{v_j \sim v_i} d_j x_j \le d_i m_i x_i; \qquad \sum_{v_j \sim v_i} \sum_{v_k \sim v_j} x_k \le d_i m_i x_i,$$

and we get

$$({q_1}^2 + bq_1)x_i \le 2d_i(d_i + m_i)x_i + 2bd_i x_i,$$

provided that $d_i + b \ge 0$.

Hence,

$${q_1}^2 + bq_1 - 2d_i(d_i + m_i + b) \le 0.$$

As $q_1 \ge 0$, we have

$$q_1 \le \frac{-b + \sqrt{b^2 + 8d_i(d_i + m_i + b)}}{2}.$$

From the inequality above, for every different values to b, we can get several distinct upper bounds. In particular, if $b = -d_i$, we get the bound (14):

$$q_1 \le \max_{v_i} \left\{ \frac{d_i + \sqrt{(d_i)^2 + 8d_i m_i}}{2} \right\}. \quad \square$$

Finally, using a technique to prove bounds on $\mu_1(G)$ due to Zhang [19], we show that (5) is also satisfied by $q_1(G)$.

Theorem 2. *Let G be a simple and connected graph. Then*

$$q_1(G) \le \max\left\{ d_i + \sqrt{d_i m_i} | v_i \in V(G) \right\}. \tag{15}$$

Moreover, if G is a k-regular, bipartite regular or semi-regular graph the equality holds.

Proof. Let x be an eigenvector corresponding to $q_1(G)$ such that $x_i \geq 0, \forall i$, and $\|x\| = 1$. So, $Q(G)x = q_1(G)x$. Moreover, for any $v_u \in V(G)$, if $q_1(G)x_u = d_u x_u + \sum_{v \sim u} x_v$,

$$q_1{}^2 x_u{}^2 = d_u{}^2 x_u{}^2 + 2 d_u x_u \sum_{v_i \sim u} x_{v_i} + \left(\sum_{v_i \sim u} x_{v_i}\right)^2. \tag{16}$$

We know that

$$\begin{aligned} 0 \leq \sum_{v_i \sim u, v_j \sim u} (x_i - x_j)^2 &\Leftrightarrow 0 \leq (d_u - 1)\sum_{v_i \sim u}(x_i)^2 - 2\sum_{v_i \sim u, v_j \sim u} x_i x_j \\ &\Leftrightarrow 0 \leq d_u \sum_{v_i \sim u}(x_i)^2 - \sum_{v_i \sim u}(x_i)^2 - 2\sum_{v_i \sim u, v_j \sim u} x_i x_j \\ &\Leftrightarrow \sum_{v_i \sim u}(x_i)^2 + 2\sum_{v_i \sim u, v_j \sim u} x_i x_j \leq d_u \sum_{v_i \sim u}(x_i)^2 \Rightarrow \left(\sum_{v_i \sim u} x_i\right)^2 \leq d_u \sum_{v_i \sim u}(x_i)^2. \end{aligned} \tag{17}$$

From (16) and (17), we get

$$q_1{}^2 x_u{}^2 \leq d_u{}^2 x_u{}^2 + 2d(u) x_u \sum_{v_i \sim u} x_i + d_u \sum_{v_i \sim u}(x_i)^2 \Leftrightarrow q_1{}^2 x_u{}^2 \leq d_u{}^2 x_u{}^2 + 2 d_u x_u (q_1 x_u - d_u x_u) + d_u \sum_{v_i \sim u}(x_i)^2.$$

Consequently,

$$\begin{aligned} \sum_{u \in V(G)} q_1{}^2 x_u{}^2 &\leq \sum_{u \in V(G)} (-d_u{}^2 + 2 d_u q_1) x_u{}^2 + \sum_{u \in V(G)} d_u \sum_{v_i \sim u} (x_i)^2 \\ &= \sum_{u \in V(G)} (2 d_u q_1 - d_u{}^2) x_u{}^2 + \sum_{u \in V(G)} d_u m_u (x_u)^2 \\ &\Leftrightarrow \sum_{u \in V(G)} q_1{}^2 x_u{}^2 \leq \sum_{u \in V(G)} (2 d_u q_1 - d_u{}^2 + d_u m_u) x_u{}^2 \\ &\Leftrightarrow \sum_{u \in V(G)} (q_1{}^2 - 2 d_u q_1 + d_u{}^2 - d_u m_u x_u{}^2) \leq 0. \end{aligned}$$

Then, there must be a vertex u, such that

$$q_1{}^2 - 2 d_u q_1 + d_u{}^2 - d_u m_u \leq 0.$$

As $q_1 \geq 0$, we have

$$q_1(G) \leq d_u + \sqrt{d_u m_u}$$

and, so

$$q_1 \leq \max\left\{ d_i + \sqrt{d_i m_i} | v_i \in V(G) \right\}. \tag{18}$$

If G is bipartite regular or semi-regular, from [19], the equality (18) holds.

Now, suppose that G is a k-regular graph. So, $\forall v_i \in V(G)$, $d_i = m_i = k$. Then, $d_i + \sqrt{d_i m_i} = 2k$ and $q_1 = 2k$. □

Let us now consider the second series of bounds (8)–(12). We first note that bound (8) was shown to hold also for $q_1(G)$ by several researchers, see e.g. [6]. Moreover, Tan, Guo and Qi [17] proved that the bound (12) is valid for $q_1(G)$. It is easy to show that the proofs of the three remaining bounds can be slightly modified to show that they apply to $q_1(G)$ in addition to $\mu_1(G)$. Recall that the line graph L_G of graph G has vertices corresponding to all edges of G and edges corresponding to pairs of incident edges of G. We then have the following property [14]:

$$\mu_1(G) \leq q_1(G) = 2 + \rho(L_G)$$

where $\rho(L_G)$ denotes the index of the line graph of G.

To find the upper bounds (9)–(11) on $\mu_1(G)$, Li and Zhang [13,19] compute in fact bounds on $\rho(L_G) = q_1(G) - 2$ using the above result. Therefore the fact that the bounds obtained for $\mu_1(G)$ are valid for $q_1(G)$ is embedded in their proofs.

3. Bounds on $\mu_1(G)$ that do not hold for $q_1(G)$

This section is dedicated to upper bounds on $\mu_1(G)$, given as functions of the degrees and the number of common neighbors of pairs of vertices, which are not valid for $q_1(G)$. They are expressed in [9,11,16] as follows:

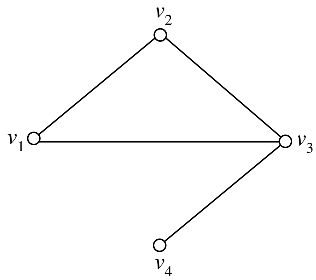

Fig. 1. Graph G.

Proposition 3. *Let G be a simple and connected graph. Then,*

$$\mu_1(G) \le \max\{d_i + d_j - |N_i \cap N_j| : 1 \le v_i < v_j \le n\},$$

where $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

Proposition 4. *Let G be a simple and connected graph. Then,*

$$\mu_1(G) \le \max\{d_i + d_j - |N_i \cap N_j| : 1 \le v_i < v_j \le n,\ v_iv_j \in E\},$$

where $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

Proposition 5. *Let G be a simple and connected graph. Then,*

$$\mu_1(G) \le \max\left\{\sqrt{2d_i(d_i + m'_i)} : v_i \in V\right\},$$

where $m'_i = \frac{\sum_{v_iv_j \in E}(d_j - |N_i \cap N_j|)}{d_i}$ and $|N_i \cap N_j|$ is the number of common neighbors of v_i and v_j.

Proposition 6. *Let G be a simple and connected graph. Then,*

$$\mu_1(G) \le \max\left\{\frac{d_i + \sqrt{(d_i)^2 + 8d_im'_i}}{2} : v_i \in V\right\},$$

where $m'_i = \frac{\sum_{v_iv_j \in E}(d_j - |N_i \cap N_j|)}{d_i}$. Moreover, the equality holds if and only if G is a bipartite regular graph.

The graph G displayed in Fig. 1 suffices to show that all bounds on the index of the Laplacian matrix given in Propositions 3–6 do not hold for the index of the signless Laplacian. Indeed,

- $q_1(G) = 4, 5616$;
- $\max\{d_i + d_j - |N_i \cap N_j| : 1 \le v_i < v_j \le n\} = 4$;
- $\max\{d_i + d_j - |N_i \cap N_j| : 1 \le v_i < v_j \le n,\ \{v_i, v_j\} \in E\} = 4$;
- $\max\left\{\sqrt{2d_i(d_i + m'_i)} : v_i \in V\right\} = 4, 4721$, and finally,
- $\max\left\{\frac{d_i + \sqrt{(d_i)^2 + 8d_im'_i}}{2} : v_i \in V\right\} = 4, 3723$.

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